October Math Gems

PROBLEM OF THE WEEK 6

§1 Answers

Problem 1.1. Solve the following equation

$$(\sqrt{2+\sqrt{3}})^x + (\sqrt{2-\sqrt{3}})^x = 2^x$$

Solution. Divide the both sides by 2^x,

$$(\frac{\sqrt{2+\sqrt{3}}}{2})^x + (\frac{\sqrt{2-\sqrt{3}}}{2})^x = 1$$

As

$$\frac{\sqrt{2+\sqrt{3}}}{2} = \cos(15) \qquad \text{and} \qquad \frac{\sqrt{2-\sqrt{3}}}{2} = \sin(15)$$

So,

$$(\cos 15)^x + (\sin 15)^x = 1 \implies x = 2$$

Using the identity,

$$\cos^2(x) + \sin^2(x) = 1$$

□

Problem 1.2. Solve for x

$$\frac{4^{2x} + 4^x + 1}{2^{2x} + 2^x + 1} = 13$$

Solution. Multiply both sides by $2^{2x} + 2^x + 1$,

$$\frac{4^{2x} + 4^x + 1}{2^{2x} + 2^x + 1} \times 2^{2x} + 2^x + 1 = 13 \times (2^{2x} + 2^x + 1)$$

$$4^{2x} + 4^x + 1 = 13 \times (2^{2x} + 2^x + 1)$$

$$2^{2\times 2x} + 2^{2\times x} + 1 = 13 \times (2^{2x} + 2^x + 1)$$

$$(2^x)^4 + (2^x)^2 + 1 = 13 \times ((2^x)^2 + 2^x + 1)$$

Let $2^x = y$,

$$(y)^4 + (y)^2 + 1 = 13 \times ((y)^2 + y + 1)$$

$$y^4 + y^2 + 1 - 13 - 13y - 13y^2 = 0$$

$$y^4 - 12y^2 - 13y - 12 = 0$$

$$(y+3)(y-4)(y^2+y+1) = 0 \implies y = -3 \qquad \text{or} \qquad y = 4$$

As $2^x = y$. So,

$$2^x = 4 \implies x = 2$$

□

Problem 1.3. If $\sin A + \sin^2 A = 1$ and $a\cos^{12} A + b\cos^8 A + c\cos^6 A - 1 = 0$, then the value of

$$b + \frac{c}{a} + b$$

is?

Solution. As $\sin^2 A + \cos^2 A = 1 \implies \sin A = \cos^2 A \qquad$ (from the givens)

So,

$$\sin^2 A = 1 - \cos^2 A = \cos^4 A \implies \cos^4 + \cos^2 = 1$$

$$(\cos^4 A + \cos^2 A)^3 = 1^3 = 1$$

Using the identity

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\cos^{12} A + 3 \times \cos^8 A \times \cos^2 A + 3 \times \cos^4 A \times \cos^4 A + \cos^6 A = 1$$

$$\cos^{12} A + 3 \times \cos^8 A \times \cos^2 A + 3 \times \cos^8 A + \cos^6 A = 1$$

$$[\text{ As } \cos^2 A = 1 - \cos^4 A]$$

$$\cos^{12} A + 3\cos^8 A \times (1 - \cos^4 A) + 3 \times \cos^8 A + \cos^6 A = 1$$

$$\cos^{12} A + 3\cos^8 A - 3\cos^{12} A + 3\cos^8 A + \cos^6 = 1$$

$$-2\cos^{12} A + 6\cos^8 A + \cos^6 A - 1 = 0 \implies a = -2, \quad b = 6, \quad c = 1$$

$$b + \frac{c}{a} + b = 6 + \frac{1}{-2} + 6 = 11.5$$

□

Problem 1.4. If $x + \frac{1}{x} = 1$, Find the value of

$$x^{21} + x^{18} + x^{12} + x^9 + x^3 + 1$$

Solution.

$$x + \frac{1}{x} = \frac{x^2}{x} + \frac{1}{x} = \frac{x^2+1}{x} = 1 \implies x^2 + 1 = x$$

Using the identity

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$x^3 + 1 = (x+1)(x^2 - x + 1) = (x+1)(0) = 0$$

$$[\text{ As } x^2 + 1 = x \implies x^2 - x + 1 = 0]$$

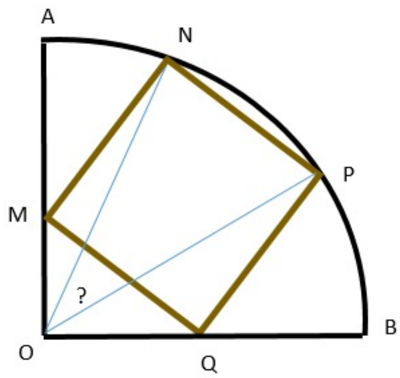
So,

$$x^3 + 1 = 0 \implies x^3 = -1$$

$$x^{21} + x^{18} + x^{12} + x^9 + x^3 + 1 = (-1)^7 + (-1)^6 + (-1)^3 + (-1) + 1 = -1$$

□

Problem 1.5. AOB is a quadrant and $MNPQ$ is a square . Find the value of the unknown angle.



Solution. Draw OK. So,

$$\begin{cases} \triangle ORQ \cong \triangle ORM \implies \angle ROQ = \angle ROM = 45 \\ NK = KP = RQ = RM = OR \\ \angle PKR = 90 \\ \angle QRK = 90 \end{cases}$$

Suppose that $OR = x$ and the radius of the circle $= 1$. So,

$$\begin{cases} RK = 2x \\ NK = x \\ NO = 1 \end{cases}$$

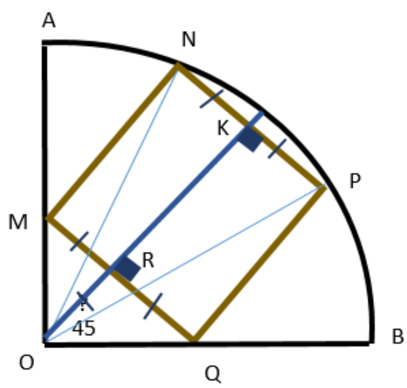
Apply Pythagoras theorem in $\triangle NKO$

$$1^2 = x^2 + 9x^2 \implies x = \frac{1}{\sqrt{10}}$$

$$\sin(\frac{1}{2}\angle NOP) = \frac{1}{\sqrt{10}}$$

$$\angle NOP = \sin^{-1}(\frac{1}{\sqrt{10}}) \times 2 = 3652'11.63''$$

□



Problem 1.6. Solve for x

$$\frac{1}{1-\sqrt{1-x}} - \frac{1}{1+\sqrt{1-x}} = \frac{\sqrt{3}}{x}$$

Solution.

$$\frac{(1+\sqrt{1-x})-(1-\sqrt{1-x})}{(1-\sqrt{1-x})(1+\sqrt{1-x})} = \frac{1+\sqrt{1-x}-1+\sqrt{1-x}}{1-1+x} = \frac{2\sqrt{1-x}}{x} = \frac{\sqrt{3}}{x}$$

$$2\sqrt{1-x} = \sqrt{3} \implies 1-x = \frac{3}{4}$$

$$x = 1-\frac{3}{4} = \frac{1}{4}$$

□

Problem 1.7. Prove or disprove: $\forall$ integer values of x, then $x^9-6x^7+9x^5-4x^3$ is divisible by 8640.

Solution. Suppose that

$$N = x^9-6x^7+9x^5-4x^3 = x^3(x^6-6x^4+9x^2-4) = x^3(x^2-1)^2(x^2-4)$$

We can write N in many ways. As

$$N = [(x-2)(x-1)x][(x-1)x(x+1)][x(x+1)(x+2)] \rightarrow (1)$$

$$N = [(x-2)(x-1)x(x+1)(x+2)][(x-1)x][x(x+1)] \rightarrow (2)$$

$$N = [(x-2)(x-1)x(x+1)][(x-1)x(x+1)(x+2)] \rightarrow (3)$$

As we see,

$$\text{in (1) } N \text{ is divisible by } 3^3 = 27$$

$$\text{in (2) } N \text{ is divisible by } 5$$

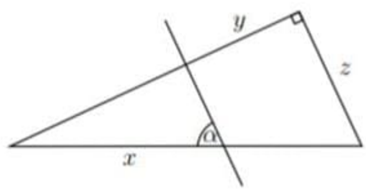
$$\text{in (3) } N \text{ is divisible by } 2^3 \implies \therefore N \text{ is divisible by } 2^6$$

Thus we can say that

$$N \text{ is divisible by } (3^3 \times 5 \times 2^6 = 8640)$$

Hence we proved it! □

Problem 1.8. The drawing below shows a right-angled triangle. A straight line crosses the triangle parallel to the line z and encloses an angle of α. The lengths x and y of the bottom and top line segments as well as the angle α are given. Find an equation for the length z.



Solution.

$$\sin\alpha = \frac{BE}{x}$$

$$BE = x \times \sin\alpha$$

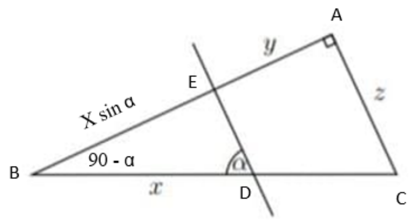
$$\angle B = 90 - \alpha$$

$$\tan(90-\alpha) = \frac{z}{x\sin\alpha + y}$$

$$\cot\alpha = \frac{z}{x\sin\alpha + y} \implies z = \cot\alpha \times (x\sin\alpha + y)$$

$$z = \frac{\cos\alpha}{\sin\alpha} \times (x\sin\alpha + y) = x\cos\alpha + y\cot\alpha$$

□



Problem 1.9. Solve this system of equations

$$\begin{cases} \sqrt{x+y} = 72 - x - y \\ \sqrt{x-y} = x - y - 30 \end{cases}$$

Solution. Let's work on the first equation:

$$\sqrt{x+y} = 72 - (x+y)$$

$$\sqrt{x+y} = 72 - (\sqrt{x+y})^2$$

$$(\sqrt{x+y})^2 + \sqrt{x+y} = 72$$

$$(\sqrt{x+y})^2 + \sqrt{x+y} + (0.5)^2 = 72 + (0.5)^2 = 72.25 = (8.5)^2$$

$$(\sqrt{x+y} + 0.5)^2 = (8.5)^2 \implies x + y = 81 \qquad \text{or} \qquad x + y = 64$$

Verify the solutions:

$$\sqrt{81} = 72 - (81) \rightarrow \text{(false)}$$

$$\sqrt{64} = 72 - (64) \rightarrow \text{(true)}$$

Now, we get

$$x + y = 64 \rightarrow (1)$$

Now, let's work on the second one:

$$\sqrt{x-y} = (\sqrt{x-y})^2 - 30$$

$$(\sqrt{x-y})^2 - \sqrt{x-y} + (0.5)^2 = 30 + (0.5)^2 = (5.5)^2$$

$$(\sqrt{x-y} - 0.5)^2 = (5.5)^2 \implies x - y = 25 \qquad \text{or} \qquad x - y = 36$$

Verify the solutions:

$$\sqrt{25} = 25 - (30) \to \text{(false)}$$
$$\sqrt{36} = 36 - (30) \to \text{(true)}$$

Now, we get

$$x - y = 36 \to (2)$$

From (1) and (2),

$$\begin{cases} x + y = 64 \\ x - y = 36 \end{cases} \implies x = 50 \qquad y = 14$$

□

Problem 1.10. Find the value of x

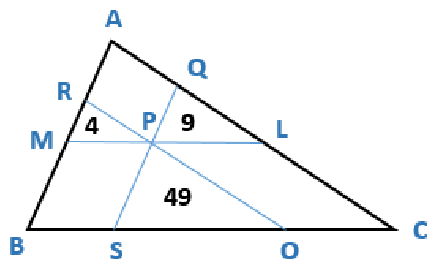
$$x^{x^6} = \sqrt{2}^{\sqrt{2}}$$

Solution.

$$(x^{x^6})^6 = (\sqrt{2}^{\sqrt{2}})^6$$
$$(x^6)^{x^6} = (\sqrt{2}^3)^{2\sqrt{2}} \implies x^6 = 2\sqrt{2} \implies x = \sqrt[4]{2}$$

□

Problem 1.11. P is in the interior of $\triangle ABC$, lines trough P parallel to the sides of $\triangle ABC$, the resulting smaller triangles have areas $t_1 = 4, t_2 = 9, t_3 = 49$. Find the area of $\triangle ABC$.



Solution. We will use the similarity of an area:
The ratio of the area of two similar triangles is equal to the square of the ratio of any pair of the corresponding sides of the similar triangles.

$$\text{in } \triangle\ LQP, OPS \begin{cases} \angle LQP = \angle OPS \\ \angle LPQ = \angle OSP \end{cases} \implies \therefore \triangle LQP \sim OPS$$

$$(\frac{QP}{PS})^2 = \frac{9}{49} \implies \frac{QP}{PS} = \frac{3}{7}$$

We can say that

$$\frac{QP}{QS} = \frac{3}{10}$$

$$\text{in } \triangle\ LQP, QSC \begin{cases} \angle LQP \qquad (\text{ is a common angle }) \\ \angle QPL = \angle QSC \end{cases} \implies \therefore \triangle LQP \sim CQS$$

$$(\frac{QP}{QS})^2 = \frac{9}{100} = \frac{\text{area of } \triangle LQP}{\text{area of } \triangle SQC} \implies \text{area of } \triangle SQC = 100$$

So, the area of the parallelogram $LPOC = 100 - (9 + 49) = 42$. As we can see that we can get the area of the parallelogram $LPOC$ in another way $(\sqrt{9}+\sqrt{49})^2-(49+9) = 42$.

As a result of this, we can get the area of another two parallelograms $ARPQ$ and $MPSB$ in the same way

$$\text{area of parallelogram } ARPQ \ = (\sqrt{4} + \sqrt{9})^2 - (4 + 9) = 12$$

$$\text{area of parallelogram } MPSB \ = (\sqrt{4} + \sqrt{49})^2 - (4 + 49) = 28$$

So,

$$\text{area of } \triangle ABC \ = 4 + 9 + 49 + 42 + 12 + 28 = 144$$

□

Problem 1.12. Solve this system of equations

$$\begin{cases} x^2 = y^3 + 1 \\ y^2 = x^3 - 23 \end{cases}$$

Solution.

$$x^2 = y^3 + 1 \implies x = \pm\sqrt{y^3 + 1}$$

Now, plugging $x = \sqrt{y^3 + 1}$ in $y^2 = x^3 - 23$

$$y^2 = (\sqrt{y^3 + 1})^3 - 23 \implies y = 2$$

$$y^2 = (y^3 + 1)^{\frac{3}{2}} - 23 \implies (y^2 + 23)^2 = (y^3 + 1)^3$$

$$y^4 + 46y^2 + 529 = (y^3 + 1)^3$$

$$(y - 2)(y^8 + 2y^7 + 4y^6 + 11y^5 + 22y^4 + 43y^3 + 89y^2 + 132y + 264) = 0$$

$$\begin{cases} y - 2 = 0 \implies y = 2 \\ (y^8 + 2y^7 + 4y^6 + 11y^5 + 22y^4 + 43y^3 + 89y^2 + 132y + 264) = 0 \qquad \text{has no solution} \in R \end{cases}$$

For $x = -\sqrt{y^3 + 1}$ in $y^2 = x^3 - 23$, there is no solution $\in R$.
Now, for getting x substitute with y in one of these equations:

$$x^2 = 8 + 1 \implies x = 3$$

$$x = 3 \qquad y = 2$$

□

Problem 1.13. Solve for x

$$(x + 2)^2 + (x + 3)^3 + (x + 4)^4 = 2$$

Solution.

$$(x+2)^2 + (x+3)^3 + (x+4)^4 - 2 = 0$$

$$(x+2)^2 - 1 + (x+3)^3 + (x+4)^4 - 1 = 0$$

$$(x^2 + 4x + 4 - 1) + (x+3)^3 + ((x+4)^4 - 1) = 0$$

$$(x+3)(x+1) + (x+3)^3 + ((x+3)(x+5)(x^2 + 8x + 17) = 0$$

$$(x+3)(x+5)(x^2 + 9x + 19) = 0$$

$$x \in \{-3, -5, \frac{-9 \pm \sqrt{5}}{2}\}$$

□

Problem 1.14. Find a, b

$$x^4 - 4x^3 + ax^2 + bx + 1 = 0$$

Solution. From the properties of the 4^{th} degree equation: the sum of the four roots is equal to the $-ve$ coefficient of x^3, and the product of the four roots is equal to absolute term. So, Suppose that $\alpha, \beta, \gamma, \delta$ are the four roots of this equation

$$\alpha + \beta + \gamma + \delta = 4$$

$$\alpha \times \beta \times \gamma \times \delta = 1$$

the arithmetic mean of the four roots is

$$\frac{\alpha + \beta + \gamma + \delta}{4} = 1$$

and the geometric mean of the four roots is

$$\sqrt[4]{\alpha \times \beta \times \gamma \times \delta} = 1$$

And as the geometric mean = arithmetic mean. So, the four values of these roots are equal. So,

$$\alpha = \beta = \gamma = \delta = 1$$

$$a = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 6 \times 1 = 6$$

$$-b = \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 4 \times 1 = 4 \implies b = -4$$

□

Problem 1.15.

$$8^a = 27^b = 125^c = 30, \qquad \frac{abc}{ab + bc + ca} = ?$$

Solution. We can solve this problem using algorithms and we will solve it easily by substituting

$$a = \frac{\log(30)}{3\log(2)} \qquad b = \frac{\log(30)}{3\log(3)} \qquad c = \frac{\log(30)}{3\log(5)}$$

But we can solve it using another method

$$8^a = 30 \implies 8 = \sqrt[a]{30}$$

$$27^b = 30 \implies 27 = \sqrt[b]{30}$$

$$125^c = 30 \implies 125 = \sqrt[c]{30}$$
$$8 \times 27 \times 125 = 30^{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$
$$(30)^3 = 30^{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \implies \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 3$$

So,

$$\frac{abc}{ab+bc+ca} = \frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} = \frac{1}{3}$$

□

Problem 1.16. If $x - 5\sqrt{x} - 1 = 0$, Find the value of

$$x^2 + \frac{1}{x^2}$$

Solution.

$$x - 1 = 5\sqrt{x}$$

Now, divide the both sides by $\sqrt{x}$

$$\sqrt{x} - \frac{1}{\sqrt{x}} = 5$$

Now, square both sides

$$(\sqrt{x} - \frac{1}{\sqrt{x}})^2 = x + \frac{1}{x} - 2 = 25 \implies x + \frac{1}{x} = 27$$

Now, squaring again both sides

$$(x + \frac{1}{x})^2 = (27)^2 = x^2 + \frac{1}{x^2} + 2 = (27)^2 \implies x^2 + \frac{1}{x^2} = 727$$

□

Problem 1.17. If $a + \frac{1}{a} = 5$, Find the value of

$$\sqrt{\frac{(a^5 + a^3)(a^3 + a)}{4a^6}}$$

Solution.

$$\sqrt{\frac{a^3(a^2+1)a(a^2+1)}{4a^6}} = \sqrt{\frac{a^4(a^2+1)(a^2+1)}{4a^6}} = \sqrt{\frac{(a^2+1)(a^2+1)}{4a^2}}$$
$$a + \frac{1}{a} = \frac{a^2+1}{a} = 5 \implies a^2 + 1 = 5a$$
$$\sqrt{\frac{(a^2+1)(a^2+1)}{4a^2}} = \sqrt{\frac{25a^2}{4a^2}} = \frac{5}{2} = 2.5$$

□

Problem 1.18. If $2f(x) + f(1-x) = x^2 \forall x$, then find

$$f(x) = ?$$

Solution.

$$2f(x) + f(1-x) = x^2 \to (1)$$

If we replace x by $1-x$ we get

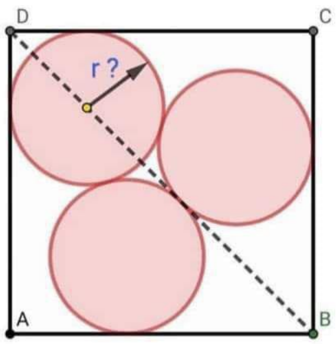
$$2f(1-x) + f(x) = (1-x)^2 \to (2)$$

Now multiply (1) by 2 and subtract it from (2)

$$3f(x) = 2x^2 - 1 - x^2 + 2x \implies f(x) = \frac{x^2 + 2x - 1}{3}$$

□

Problem 1.19. Three circles with the same radius r are inscribed in a square that has a length of 1. Find the length of the radius.



AB = 1

Solution. As

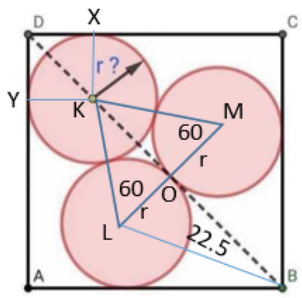
$$AB = AD = 1 \implies DB = \sqrt{2}$$

$$DB = DK + KO + OB$$

$$\begin{cases} DK = \sqrt{2}r & \text{(As } XKYD \text{ is a square)} \\ KO = 2r \times \sin(60) = \frac{\sqrt{3}}{2} \times 2r = \sqrt{3}r & \text{(As } KLM \text{ is an equilateral triangle)} \\ OB = \frac{r}{\tan(22.5)} & \text{(As } \angle OBL = \frac{45}{2} = 22.5 \text{)} \end{cases}$$

$$\sqrt{2} = \sqrt{2}r + \sqrt{3}r + \frac{r}{\tan(22.5)} \implies r = \frac{\sqrt{2}}{\sqrt{2} + \sqrt{3} + \frac{1}{\tan 22.5}} = 0.254$$

□



AB = 1

Problem 1.20.

$$(x + x^2 + x^3) + (\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}) = 28$$

Find the value of

$$(2x - 3)^2$$

Solution.

$$(x + \frac{1}{x}) + (x^2 + \frac{1}{x^2}) + (x^3 + \frac{1}{x^3}) = 28$$

$$(x + \frac{1}{x}) + ((x + \frac{1}{x})^2 - 2) + ((x + \frac{1}{x})^3 - 3(x + \frac{1}{x})) = 28$$

Let

$$x + \frac{1}{x} = a$$

$$a + a^2 - 2 + a^3 - 3a = 28 \implies a^3 + a^2 - 2a - 30 = 0$$

$$(a - 3)(a^2 + 4a + 10) = 0 \implies x + \frac{1}{x} = 3$$

$$\frac{x^2}{x} + \frac{1}{x} = 3 \implies \frac{x^2 + 1}{x} = 3 \implies x^2 - 3x = -1$$

$$(2x - 3)^2 = 4x^2 - 12x + 9 = 4(x^2 - 3x) + 9 = -4 + 9 = 5$$

$\square$